Journal of Engineering Mathematics, Vol. 6, No. 2, April 1972
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

Equivalence of a Cauchy System and a Class of Boundary-Value Problems in Thin Plate Theory

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(Received June 4, 1971)

SUMMARY
In [1] we reduced the solution of a classical boundary-value problem, namely the biharmonic equation in a rectangular domain, to a Cauchy formulation. The theory was developed in the context of elementary thin plate theory. It was shown that a rectangular plate with three edges clamped and the fourth edge free can be completely described by a system of integro-differential equations subject to initial values. In this paper we prove the converse, i.e., that any solution of the Cauchy system is a solution of the biharmonic equation, completing the equivalence.

1. Introduction

In recent publication [1] we studied the reduction of a classical elliptic boundary-value problem, namely the biharmonic equation in a rectangular domain, to a Cauchy or, initial-value formulation. The theory was developed in the context of elementary thin plate theory. Using ideas of invariant imbedding [2] it was shown that a rectangular plate with three edges clamped and the fourth edge free can be completely described by a system of integro-differential equations subject to initial values. Classical reciprocity relations were proved in the context of the invariant-imbedding theory and some applications of the fundamental solution were considered.

Interest in the reduction of boundary-value problems to initial-value formulations is stimulated by both theoretical and practical considerations. In the first place, since boundary-value problems cannot generally be solved directly by numerical methods, a reduction to an initial-value formulation, for which many standard procedures are available, is of considerable numerical interest. On the other hand, since structural perturbations are at the basis of invariant imbedding, this theory offers a natural, unified device to study the solution of many types of equations in terms of non-classical variables such as length, thickness and physical constants [3].

These, among other reasons, show the necessity of developing a rigorous approach on which to base the invariant imbedding procedures used in the reduction of boundary-value problems to initial-value formulations. This paper is a step in that direction. Using the results obtained in [1], we prove the converse i.e., that every solution of the Cauchy system satisfies the original boundary-value problem thus completing the equivalence.

2. The Problem

Consider the deflection of a thin rectangular plate clamped at the edges $x=0$, $y=0$ and $y=1$ and subject to moments, $m(y)$, normal to the free edge $x=a$ and vertical forces $n(y)$ along this edge. The deflection is given by the biharmonic equation

$$\nabla^4 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy} = 0 ,$$

subject to the boundary conditions

$$w(0, y) = w(x, 0) = w(x, 1) = 0 ,$$
$$w_x(0, y) = w_y(x, 0) = w_y(x, 1) = 0 ,$$

and

$$w_{xx}(a, y) = m(y),$$
$$w_{xxx}(a, y) + 2w_{xyy}(a, y) = n(y),$$

where we have assumed the Poisson ratio of the material to be zero.

We can express w via the superposition

$$w(x, y) = \int_0^1 v(x, y, a, \sigma)n(\sigma)d\sigma + \int_0^1 u(x, y, a, \sigma)m(\sigma)d\sigma,$$

where u and v satisfy

$$\nabla^4 u = 0 \tag{1}$$
$$\nabla^4 v = 0, \tag{2}$$

subject to

$$u_{11}(a, y, a, \sigma) = -\delta(y-\sigma), \tag{3}$$
$$u_{111}(a, y, a, \sigma) + 2u_{122}(a, y, a, \sigma) = 0, \tag{4}$$
$$v_{11}(a, y, a, \sigma) = 0, \tag{5}$$
$$v_{111}(a, y, a, \sigma) + 2v_{122}(a, y, a, \sigma) = -\delta(y-\sigma), \tag{6}$$

and

$$\begin{aligned} &u(0, y, a, \sigma) = u(x, 0, a, \sigma) = u(x, 1, a, \sigma) = 0, \\ &v(0, y, a, \sigma) = v(x, 0, a, \sigma) = v(x, 1, a, \sigma) = 0, \\ &u_1(0, y, a, \sigma) = u_2(x, 0, a, \sigma) = u_2(x, 1, a, \sigma) = 0 \\ &v_1(0, y, a, \sigma) = v_2(x, 0, a, \sigma) = v_2(x, 1, a, \sigma) = 0 \end{aligned} \tag{7}$$

for $0 < \sigma < 1$, $0 \leqq x \leqq a$, where $\delta(t)$ is the Dirac delta function. Note we have explicitly included the dependence of both u and v on the length of the plate, a, and we have adopted the notation u_i, $i = 1,2,3,4$, to denote partial differentiation with respect to the ith variable of u. We seek solutions for u and v since w can be obtained directly via the above superposition.

3. The Invariant Imbedding Method [1]

The invariant imbedding method regards the length of the plate, a, as the independent variable and keeps the value of x fixed. Hence we examine $u(x, y, a, \sigma)$ and $v(x, y, a, \sigma)$ for values of $a \geqq x$. The invariant imbedding method proceeds in two parts. First for $a \geqq 0$ we solve

$$\begin{aligned} p_1(a, y, \sigma) = \delta(y-\sigma) &+ 2\int_0^1 p(a, y, \eta)p_{22}(a, \eta, \sigma)d\eta \\ &- \int_0^1 q(a, y, \eta)r_{2222}(a, \eta, \sigma)d\eta \end{aligned} \tag{8}$$

$$\begin{aligned} q_1(a, y, \sigma) = p(a, y, \sigma) &+ 2\int_0^1 p(a, y, \eta)q_{22}(a, \eta, \sigma)d\eta \\ &- \int_0^1 q(a, y, \eta)s_{2222}(a, \eta, \sigma)d\eta, \end{aligned} \tag{9}$$

$$\begin{aligned} r_1(a, y, \sigma) = p(a, y, \sigma) &+ 2\int_0^1 r(a, y, \eta)p_{22}(a, \eta, \sigma)d\eta \\ &- \int_0^1 s(a, y, \eta)r_{2222}(a, \eta, \sigma)d\eta, \end{aligned} \tag{10}$$

$$s_1(a, y, \sigma) = q(a, y, \sigma)+r(a, y, \sigma) + 2\int_0^1 r(a, y, \eta) q_{22}(a, \eta, \sigma) d\eta - \int_0^1 s(a, y, \eta) s_{2222}(a, \eta, \sigma) d\eta , \tag{11}$$

subject to initial conditions

$$p(0, y, \sigma) = q(0, y, \sigma) = r(0, y, \sigma) = s(0, y, \sigma) = 0 , \tag{12}$$

and auxiliary conditions

$$p(a, y, \sigma) = q(a, y, \sigma) = r(a, y, \sigma) = s(a, y, \sigma) = 0 , \tag{13}$$

if $y=0, 1$ or $\sigma=0, 1$.

Then at $a=x$ we adjoin the equations

$$\theta_3(x, y, a, \sigma) = 2\int_0^1 \theta(x, y, a, \eta) p_{22}(a, \eta, \sigma) d\eta - \int_0^1 \psi(x, y, a, \eta) r_{2222}(a, \eta, \sigma) d\eta , \tag{14}$$

$$\psi_3(x, y, a, \sigma) = \theta(x, y, a, \sigma) + 2\int_0^1 \theta(x, y, a, \eta) q_{22}(a, \eta, \sigma) d\eta - \int_0^1 \psi(x, y, a, \eta) s_{2222}(a, \eta, \sigma) d\eta , \tag{15}$$

$$u_3(x, y, a, \sigma) = 2\int_0^1 u(x, y, a, \eta) p_{22}(a, \eta, \sigma) d\eta - \int_0^1 v(x, y, a, \eta) r_{2222}(a, \eta, \sigma) d\eta , \tag{16}$$

$$v_3(x, y, a, \sigma) = u(x, y, a, \sigma) + 2\int_0^1 u(x, y, a, \eta) q_{22}(a, \eta, \sigma) d\eta - \int_0^1 v(x, y, a, \eta) s_{2222}(a, \eta, \sigma) d\eta , \tag{17}$$

subject to the initial conditions

$$\theta(x, y, x, \sigma) = p(x, y, \sigma) , \tag{18}$$

$$\psi(x, y, x, \sigma) = q(x, y, \sigma) , \tag{19}$$

$$u(x, y, x, \sigma) = r(x, y, \sigma) , \tag{20}$$

$$v(x, y, x, \sigma) = s(x, y, \sigma) , \tag{21}$$

and the auxiliary conditions

$$\begin{aligned} &\theta(x, y, a, \sigma) = \psi(x, y, a, \sigma) = u(x, y, a, \sigma) = v(x, y, a, \sigma) = 0 , \\ &\theta_2(x, y, a, \sigma) = \psi_2(x, y, a, \sigma) = u_2(x, y, a, \sigma) = v_2(x, y, a, \sigma) = 0 , \end{aligned} \tag{22}$$

if $y=0, 1$. The entire set of equations, (8)–(11), (14)–(17), is then integrated to the desired value of a.

4. Validation

We will now show that any solution of (8)–(22) is a solution of the original system (1)–(7). First we will show

$$\theta(x, y, a, \sigma) = u_1(x, y, a, \sigma) \tag{23}$$

$$\psi(x, y, a, \sigma) = v_1(x, y, a, \sigma). \tag{24}$$

Differentiating (16) and (17) with respect to x, we have

$$u_{13}(x, y, a, \sigma) = 2\int_0^1 u_1(x, y, a, \eta) p_{22}(a, \eta, \sigma) d\eta, \tag{25}$$
$$- \int_0^1 v_1(x, y, a, \sigma) r_{2222}(a, \eta, \sigma) d\eta$$

and

$$v_{13}(x, y, a, \sigma) = u_1(x, y, a, \sigma) + 2\int_0^1 u_1(x, y, a, \eta) q_{22}(a, \eta, \sigma) d\eta$$
$$- \int_0^1 v_1(x, y, a, \eta) s_{2222}(a, \eta, \sigma) d\eta. \tag{26}$$

We see then by comparison with (14) and (15) that u_1 and v_1 satisfy the same differential equations as θ and ψ. Direct differentiation of (22) yields

$$u_1\ (x, y, a, \sigma) = v_1\ (x, y, a, \sigma) = 0,$$
$$u_{12}(x, y, a, \sigma) = v_{12}(x, y, a, \sigma) = 0. \tag{27}$$

if $y=0, 1$ or $\sigma=0, 1$, so that the auxiliary conditions also agree. To show that the initial conditions are the same, we start by differentiating (20) with respect to x,

$$u_1(x, y, x, \sigma) = r_1(x, y, \sigma) - u_3(x, y, x, \sigma). \tag{28}$$

Using (10) and (16), with $a=x$, in (28) we find

$$u_1(x, y, x, \sigma) = p(x, y, \sigma). \tag{29}$$

Using (21), (11) and (17) in the same manner we have

$$v_1(x, y, x, \sigma) = q(x, y, \sigma). \tag{30}$$

Since the above two relations hold for all x, the equations defining u_1 and v_1 are exactly the same as those for θ and ψ so (23) and (24) must hold.

We will now show that u and v satisfy the boundary conditions of (3)–(7). We start by taking $x=0$ in (15), (17), (20), (21) and (22). Thus, $u(0, y, a, \sigma)$ and $v(0, y, a, \sigma)$ satisfy

$$u_3(0, y, a, \sigma) = 2\int_0^1 u(0, y, a, \eta) p_{22}(a, \eta, \sigma) d\eta$$
$$- \int_0^1 v(0, y, a, \eta) s_{2222}(a, \eta, \sigma) d\eta,$$

$$v_3(0, y, a, \sigma) = u(0, y, a, \sigma) + 2\int_0^1 u(0, y, a, \eta) q_{22}(a, \eta, \sigma) d\eta$$
$$- \int_0^1 v(0, y, a, \eta) s_{2222}(a, \eta, \sigma) d\eta, \tag{31}$$

subject to

$$u(0, y, 0, \sigma) = 0,$$
$$v(0, y, 0, \sigma) = 0, \tag{32}$$

and

$$u\ (0, y, a, \sigma) = v\ (0, y, a, \sigma) = 0,$$
$$u_2(0, y, a, \sigma) = v_2(0, y, a, \sigma) = 0, \tag{33}$$

if $y=0, 1$ or $\sigma=0, 1$. Eqs. (31)–(33) define a homogeneous initial value problem with zero initial conditions, thus we must have

$$u(0, y, a, \sigma) = 0 ,$$
$$v(0, y, a, \sigma) = 0 , \tag{34}$$

for all a. The rest of the conditions of (7) follow by a similar argument.

To prove (3)–(6) we proceed as follows. We start by differentiating with respect to x.

$$u_{11}(x, y, x, \sigma) = -u_{13}(x, y, x, \sigma) + p_1(x, y, \sigma) , \tag{35}$$

and via (23)

$$u_{11}(x, y, x, \sigma) = -\theta_3(x, y, x, \sigma) + p_1(x, y, \sigma) . \tag{36}$$

Using (8) and (14) with $a = x$ in (36) we find

$$u_{11}(x, y, x, \sigma) = \delta(y - \sigma) , \tag{37}$$

which since it is valid for all x, is (3). In a similar manner, from (30), (9) and (15) we find (5) or

$$v_{11}(x, y, x, \sigma) = 0 . \tag{38}$$

To prove (4) we differentiate (37) with respect to x.

$$u_{111}(x, y, x, \sigma) = u_{113}(x, y, x, \sigma) . \tag{39}$$

Differentiating (16) twice with respect to x, we find for $a = x$,

$$u_{113}(x, y, x, \sigma) = 2\int_0^1 u_{11}(x, y, x, \eta) p_{22}(x, \eta, \sigma) d\eta - \int_0^1 v_{11}(x, y, x, \eta) r_{2222}(x, \eta, \sigma) d\eta . \tag{40}$$

then by (37) and (38)

$$u_{113}(x, y, x, \sigma) = 2p_{22}(x, y, \sigma) . \tag{41}$$

Differentiating (29) twice with respect to y we find

$$u_{122}(x, y, x, \sigma) = p_{22}(x, y, \sigma) . \tag{42}$$

Finally combining (39), (41) and (42) we have

$$u_{111}(x, y, x, \sigma) + 2u_{122}(x, y, x, \sigma) = 0 , \tag{43}$$

which is (4). Eq. (6) follows by similar reasoning.

We now have to show the u and v satisfy the biharmonic equation. We start by differentiating (43)

$$u_{1111}(x, y, x, \sigma) + 2u_{1122}(x, y, x, \sigma) = -u_{1113}(x, y, x, \sigma) - 2u_{1223}(x, y, x, \sigma) . \tag{44}$$

Differentiation of (16) yields for $a = x$

$$u_{1113}(x, y, x, \sigma) = 2\int_0^1 u_{111}(x, y, x, \eta) p_{22}(x, \eta, \sigma) d\eta - \int_0^1 v_{111}(x, y, x, \eta) r_{2222}(x, \eta, \sigma) d\eta \tag{45}$$

and

$$u_{1223}(x, y, x, \sigma) = 2\int_0^1 u_{122}(x, y, x, \eta) p_{22}(x, \eta, \sigma) d\eta - \int_0^1 v_{122}(x, y, x, \eta) r_{2222}(x, \eta, \sigma) d\eta . \tag{46}$$

Combining these two equations,

$$u_{1113}(x, y, x, \sigma)+2u_{1223}(x, y, x, \sigma) = 2\int_0^1 [u_{111}(x, y, x, \eta)+2u_{122}(x, y, x, \eta)]p_{22}(x, \eta, \sigma)d\eta$$

$$-\int_0^1 [v_{111}(x, y, x, \eta)+2v_{122}(x, y, x, \sigma)] = r_{2222}(x, \eta, \sigma)d\eta\,. \quad (47)$$

In view of the boundary conditions, (4) and (6), (47) becomes

$$u_{113}(x, y, x, \sigma)+2u_{1223}(x, y, x, \sigma) = r_{2222}(x, y, \sigma)\,. \quad (48)$$

Differentiation of (20) gives us the equation

$$u_{2222}(x, y, x, \sigma) = r_{2222}(x, y, \sigma)\,. \quad (49)$$

Combining (44), (48) and (49) we have

$$u_{1111}(x, y, x, \sigma)+2u_{1122}(x, y, x, \sigma)+u_{2222}(x, y, x, \sigma) = 0\,, \quad (50)$$

so that the biharmonic equation holds at the edge. We repeat the argument to show

$$v_{1111}(x, y, x, \sigma)+2v_{1122}(x, y, x, \sigma)+v_{2222}(x, y, x, \sigma) = 0\,. \quad (51)$$

To prove that the biharmonic equation is satisfied at internal points, we form the two functions.

$$\alpha(x, y, a, \sigma) = u_{1111}(x, y, a, \sigma)+2u_{1122}(x, y, a, \sigma)+u_{2222}(x, y, a, \sigma)\,, \quad (52)$$

$$\beta(x, y, a, \sigma) = v_{1111}(x, y, a, \sigma)+2v_{1122}(x, y, a, \sigma)+v_{2222}(x, y, a, \sigma)\,. \quad (53)$$

Clearly by (50) and (51)

$$\alpha(x, y, x, \sigma) = 0\,,$$
$$\beta(x, y, x, \sigma) = 0\,. \quad (54)$$

We will find initial value problems for α and β. Differentiation of (52) with respect to a yields

$$\alpha_3(x, y, a, \sigma) = u_{11113}(x, y, a, \sigma)+2u_{11223}(x, y, a, \sigma)+u_{22223}(x, y, a, \sigma)\,. \quad (55)$$

Each of the terms on the right of (55) can be evaluated by repeated differentiation of (16), i.e.

$$u_{11113}(x, y, a, \sigma) = 2\int_0^1 u_{1111}(x, y, a, \eta)p_{22}(a, \eta, \sigma)d\eta$$
$$-\int_0^1 v_{1111}(x, y, a, \eta)r_{2222}(a, \eta, \sigma)d\eta\,,$$

$$u_{11223}(x, y, a, \sigma) = 2\int_0^1 u_{1122}(x, y, a, \eta)p_{22}(a, \eta, \sigma)d\eta$$
$$-\int_0^1 v_{2222}(x, y, a, \eta)r_{2222}(a, \eta, \sigma)d\eta\,. \quad (56)$$

Adding these terms and using (52) and (53) we have

$$\alpha_3(x, y, a, \sigma) = 2\int_0^1 \alpha(x, y, a, \eta)p_{22}(a, \eta, \sigma)d\eta - \int_0^1 \beta(x, y, a, \eta)r_{2222}(a, \eta, \sigma)d\eta\,. \quad (57)$$

Similarly we find

$$\beta_3(x, y, a, \sigma) = \alpha(x, y, a, \sigma) + 2\int_0^1 \alpha(x, y, a, \eta)q_{22}(a, \eta, \sigma)d\eta$$
$$-\int_0^1 \beta(x, y, a, \eta)s_{2222}(a, \eta, \sigma)d\eta\,. \quad (58)$$

From (22) we find

$$\alpha(x, y, a, \sigma) = \beta(x, y, a, \sigma) = \alpha_2(x, y, a, \sigma) = \beta_2(x, y, a, \sigma) = 0\,, \quad (59)$$

if $y=0$, 1 or $\sigma=0$, 1. Thus, (54), (57), and (58) and (59) define a homogeneous initial value problem with a zero initial condition and therefore we have

$$\alpha(x, y, a, \sigma)=0\,,$$
$$\beta(x, y, a, \sigma)=0\,, \tag{60}$$

or

$$\nabla^4 u(x, y, a, \sigma)=0\,,$$
$$\nabla^4 v(x, y, a, \sigma)=0\,, \tag{61}$$

and the verification is complete.

5. Conclusion

We have shown the equivalence between a certain classical boundary-value problem and a certain Cauchy system. The practical importance of this result is that the Cauchy system can be readily treated by a number of standard numerical techniques, as opposed to boundary-value problems which are not in general amenable to direct numerical treatment. At the same time, a number of theoretical advantages are associated with Cauchy formulations, especially those related to the study of semi-group properties in terms of non-classical imbedding variables such as length, thickness or physical constants. Future papers will deal with the details of numerical procedures.

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